VI Linear Differential Equations of first order

A differential equation is said to be linear if the dependent variable and its derivatives appear only in the first degree.

In other words, no term involves powers of derivatives or powers of dependent variables or product of derivatives and / or dependent variables.

A differential equation of the form $\boxed{\frac{dy}{dx} + Py = Q.}$ — ①

Where P, Q are functions of x or constant, is called a linear differential equations of the first order.

Method of solution :

An I.F. of equation (1) is $e^{\int P dx}$ ie $I.F. = e^{\int P dx}$

Therefore G. S. is given by

$$y \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} dx + C$$

similarly,

$$\frac{dx}{dy} + Px = Q. \quad - ②$$

Where P and Q are functions of y or constants, is called a linear differential equation of the first order.

Method of solution : We write I.F. of equation ② as $\boxed{I.F. = e^{\int P dy}}$ and G.S. is given by

$$x \cdot e^{\int P dy} = \int Q \cdot e^{\int P dy} \cdot dy + C$$

Note: The coefficient of $\frac{dy}{dx}$ (or $\frac{dx}{dy}$) in the linear differential equation must be one.

Illustrations :

① solve $(1+y^2) + (x - e^{-\tan^{-1}y}) \frac{dy}{dx} = 0$

Sol : It can be written as $(1+y^2) \frac{dx}{dy} + x = e^{-\tan^{-1}y}$

ie $\frac{dx}{dy} + \frac{1}{1+y^2} x = \frac{e^{-\tan^{-1}y}}{1+y^2}$

which is linear in x. $\therefore$ I.F. $= e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$

$\therefore$ G.S. is $x \cdot e^{\tan^{-1} y} = \int \frac{e^{-\tan^{-1} y}}{1+y^2} \cdot e^{\tan^{-1} y} dy + c$

$$= \int \frac{dy}{1+y^2} + c$$

$$\boxed{x \cdot e^{\tan^{-1} y} = \tan^{-1} y + c}$$

2) Solve $x^2(x^2-1) \frac{dy}{dx} + x(x^2+1) y = x^2 - 1$

Sol) $\frac{dy}{dx} + \frac{x^2+1}{x(x^2-1)} \cdot y = \frac{1}{x^2}$ Which is linear in y

with $p = \frac{x^2+1}{x(x^2-1)}$ & $Q = \frac{1}{x^2}$ &

$$\int p dx = \int \frac{x^2+1}{x(x^2-1)} dx = \int \left(\frac{1}{x-1} + \frac{1}{x+1} - \frac{1}{x} \right) dx$$

$$= \log(x-1) + \log(x+1) - \log x$$

$$= \log \left(\frac{x^2-1}{x} \right)$$

I.F. $= e^{\int p dx} = \frac{x^2-1}{x}$

G.S. is $y \cdot \left(\frac{x^2-1}{x} \right) = \int \frac{1}{x^2} \left(\frac{x^2-1}{x} \right) dx + c$

$$= \int \left(\frac{1}{x} - \frac{1}{x^3} \right) dx + c$$

$$= \log x + \frac{1}{2x^2} + c$$

ie $\frac{y(x^2-1)}{x} - \log x - \frac{1}{2x^2} = c$ is the required G.S.

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(3) solve $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$ or $\frac{dy}{dx} + \left(\frac{1}{\sqrt{x}}\right)y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

Here $P = \frac{1}{\sqrt{x}}$, $Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$ $\therefore$ I.F. $= e^{\int \frac{1}{\sqrt{x}}dx} = e^{2\sqrt{x}}$

G.S. is $y \cdot e^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} e^{2\sqrt{x}} dx + c$

$= \int x^{-\frac{1}{2}} dx + c$

$\boxed{y e^{2\sqrt{x}} = 2\sqrt{x} + c}$ is G.S.

(4) Solve $(1+\sin y)\frac{dx}{dy} = 2y\cos y - x(\sec y + \tan y)$

Dividing by $(1+\sin y)$

$\frac{dx}{dy} + \left(\frac{\sec y + \tan y}{1+\sin y}\right) \cdot x = \frac{2y\cos y}{1+\sin y}$

ie $\frac{dx}{dy} + \left(\frac{\frac{1+\sin y}{\cos y}}{1+\sin y}\right)x = \frac{2y\cos y}{1+\sin y}$

$\frac{dx}{dy} + (\sec y)x = \frac{2y\cos y}{1+\sin y}$

Which is linear in x with $P = \sec y$, $Q = \frac{2y\cos y}{1+\sin y}$

I.F. $= e^{\int \sec y\, dy} = e^{\log(\sec y + \tan y)} = \sec y + \tan y$

$\therefore$ G.S. $x(\sec y + \tan y) = \int \frac{2y \cdot \cos y}{1+\sin y} \cdot (\sec y + \tan y)\, dy + c$

$= \int \left(\frac{2y\cos y}{1+\sin y}\right)\left(\frac{1+\sin y}{\cos y}\right) dy + c$

ie $x(\sec y + \tan y) = \int 2y\, dy + c = y^2 + c$ is G.S.

5. Solve $y^2 + \left(x - \frac{1}{y}\right)\frac{dy}{dx} = 0$

Note that the equation contains y^2 and so it cannot be linear in y. We try to see whether it is linear in x. We write the given eqn as

$$y^2 \frac{dx}{dy} + x = \frac{1}{y} \Rightarrow \frac{dx}{dy} + \frac{1}{y^2}x = \frac{1}{y^3} \quad \text{(linear in } x\text{)}$$

$$I.F = e^{\int \frac{1}{y^2} dy} = e^{-1/y}$$

G.S is $\quad x \cdot e^{-\frac{1}{y}} = \int \frac{1}{y^3} e^{-\frac{1}{y}} dy + C = \int \frac{1}{y} e^{-\frac{1}{y}}\left(\frac{1}{y^2}dy\right) + C$

$$= \int (-t)\, e^t\, dt + C \qquad \left(\because t = \frac{-1}{y}\right)$$

$$= -(t e^t - e^t) + C$$

$$= e^{-\frac{1}{y}}\left(1 + \frac{1}{y}\right) + C$$

$\boxed{x = 1 + \frac{1}{y} + C e^{\frac{1}{y}}}$ is the required G.S.

Solve the following linear differential equations

Ex.

1) $\sin x \cdot \frac{dy}{dx} + 2y = \tan^3 \frac{x}{2}$ Ans: $y \cdot \tan^2 \frac{x}{2} = \frac{\tan^5 \frac{x}{2}}{5} + C$

2) $\cosh x \cdot \frac{dy}{dx} + y \cdot \sinh x = 2\cosh^2 x \cdot \sinh x$

3) $x^2 \frac{dy}{dx} = 3x^2 - 2xy + 1$ Ans: $yx^2 = x^3 + x + C$

4. $(x^2+1)\frac{dy}{dx} + 4xy = \frac{1}{(x^2+1)^2}$ Ans: $y(x^2+1)^2 = \tan^{-1}x + C$

5. $(x + 2y^3)\frac{dy}{dx} = y$ Ans: $x = y^3 + Cy$

$(e^{-y}\sec^2 y - x)\,dy = dx$ Ans: $xe^y = C + \tan y$

7) $(1+x^2)\,dy = (\tan^{-1}x - y)\,dx$ Ans: $y = \tan^{-1}x - 1 + Ce^{-\tan^{-1}x}$

8) $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$ Ans: $x = \tan^{-1}y - 1 + Ce^{-\tan^{-1}y}$

9) $\sqrt{a^2+x^2}\,\frac{dy}{dx} + y = \sqrt{a^2+x^2} - x$ Ans: $y(x+\sqrt{a^2+x^2}) = a^2\log(x + \sqrt{a^2+x^2}) + C$

10) $ye^{y} = (y^3 + 2xe^{y})\frac{dy}{dx}$ Ans: $\frac{x}{y^2} + e^{-y} = C$

11) $\frac{dy}{dx} = \frac{e^x - 3xy}{x^2}$ Ans: $x^3y = C + (x-1)e^x$

12) $x(1-x^2)\frac{dy}{dx} + (2x^2-1)y = x^3$ Ans: $y = x + Cx\sqrt{1-x^2}$

13) $(1+x^2)\frac{dy}{dx} + xy = 1$.

14) $(1-x^2)\frac{dy}{dx} = 1 + xy$ Ans: $y\sqrt{1-x^2} = \sin^{-1}x + C$

15) $\frac{dy}{dx} + y.\cot x = \sin 2x$ Ans: $y = \frac{2}{3}\sin^3 x + C\,\mathrm{cosec}\,x$

16) $(2y + x^2)\,dx = x\,dy$ Ans: $y = x^2\log(cx)$

17) $\frac{dy}{dx} + \frac{y}{1-x} = x^2 - x$ Ans: $2y = (1-x)(C_1 - x^2)$

18) $\frac{dy}{dx} + (1+2x)y = e^{-x^2}$ Ans: $y.e^{x^2+x} = e^x + C$

19) $(x^2+1)\frac{dy}{dx} = x^3 - 2xy + x$ Ans: $y(x^2+1) = \frac{x^4}{4} + \frac{x^2}{2} + C$

20) $\cos\frac{dy}{dx} + y = \sin x$ Ans: $y = 1 + (C - x)(\sec x - \tan x)$

21) $x\cos x\frac{dy}{dx} + (\cos x - x\sin x)y = 1$ Ans: $xy\cos x - x = C$

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(22) $$\frac{dy}{dx} + \frac{y}{(1-x)\sqrt{x}} = 1 - \sqrt{x}$$

Ans: $y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = x + \frac{2}{3}x^{3/2} + c$

Equations Reducible to the linear Form

① **Bernoulli's differential equation** A differential equation of the form

$$\frac{dy}{dx} + Py = Q \cdot y^n$$

is called as Bernoulli's differential equation.

Method of solution: we divide by y^n.

$\therefore\ y^{-n}\frac{dy}{dx} + P y^{1-n} = Q$, put $y^{1-n} = u$

$\therefore\ (1-n)\, y^{-n}\frac{dy}{dx} = \frac{du}{dx}$

$$\therefore\ \frac{1}{1-n}\frac{du}{dx} + Pu = Q$$

or $$\frac{du}{dx} + (1-n)P \cdot u = (1-n)Q$$

Which is linear and therefore, can be solved by using method discussed in type VI.

② Similarly $\boxed{\frac{dx}{dy} + Px = Q \cdot x^n}$ is also called as Bernoulli's differential equation. Now dividing by x^n

$x^{-n}\frac{dx}{dy} + P\,x^{1-n} = Q.$ ~~put $x^{1-n} = u$~~

put $x^{1-n} = u$ $\therefore\ (1-n)\,x^{-n}\frac{dx}{dy} = \frac{du}{dy}$

$\frac{du}{dy} + (1-n)Pu = (1-n)Q.$ Which is linear & therefore can be solved.

③ Equation of the form $\boxed{f'(y)\frac{dy}{dx}+Pf(y)=Q}$ are at once reducible to the linear form by the substitution of $f(y)=u$, & $f'(y)\frac{dy}{dx}=\frac{du}{dx}$ transforming the equation to

$$\frac{du}{dx}+Pu=Q \text{, Which is linear}$$

4) similarly for $\boxed{f'(x)\frac{dx}{dy}+P\,f(x)=Q}$, We substitute $f(x)=u$ & $f'(x)\frac{dx}{dy}=\frac{dy}{dy}$ ∴ Equation reduces to

$$\boxed{\frac{du}{dx}+Pu=Q} \quad \text{Which is linear.}$$

Examples

① solve $\sin y\frac{dy}{dx}=\cos x\,(2\cos y-\sin^2 x)$

Here we will use the substitution $\cos y=u$,

∴ $\sin y\cdot\frac{dy}{dx}=-\frac{du}{dx}$ and equation reduces to

$$-\frac{du}{dx}=2u\cos x-\sin^2 x\cdot\cos x$$

or $\frac{du}{dx}+(2\cos x)u=\sin^2 x\cdot\cos x$. Which is linear in u

Here $P=2\cos x$, $Q=\sin^2 x\cdot\cos x$.

$$I.F=e^{\int Pdx}=e^{\int 2\cos x\,dx}=e^{2\sin x}$$

G.S is $ue^{2\sin x}=\int\sin^2 x\cdot\cos x\,e^{2\sin x}\cdot dx+C$

$$=\int e^{2t}\cdot t^2\,dt+c \qquad (\because \sin x=t)$$

$$=t^2\left(\frac{e^{2t}}{2}\right)-(2t)\left(\frac{e^{2t}}{4}\right)+(2)\left(\frac{e^{2t}}{8}\right)+c$$

$$u \cdot e^{2\sin x} = \frac{e^{2t}}{4}(2t^2 - 2t + 1) + C$$

$$u = \frac{1}{4}(2\sin^2 x - 2\sin x + 1) + C \cdot e^{-2\sin x}$$

or $$4 \cdot \cos y = 2\sin^2 x - 2\sin x + 1 + C_1 e^{-2\sin x}$$ is G.S.

Ex (2) solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x) e^x \sec y$

we divide by $\sec y$. $\therefore \cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x$

which can be reduced to linear form by putting

$\sin y = u$ $\quad \therefore \cos y \frac{dy}{dx} = \frac{du}{dx}$

$\frac{du}{dx} - \frac{u}{1+x} = (1+x)e^x$ which is linear in u

with $P = -\frac{1}{1+x}$, $Q = (1+x)e^x$

$$I.F. = e^{-\int \frac{1}{1+x} dx} = e^{-\log(1+x)} = \frac{1}{1+x}$$

$\therefore$ G.S. is

$$u \cdot \left(\frac{1}{1+x}\right) = \int e^x (1+x) \frac{1}{1+x} dx + C$$

$$\Rightarrow \frac{u}{1+x} = e^x + c$$

$$\Rightarrow \boxed{\sin y = e^x(1+x) + c(1+x)}$$ is G.S.

Ex (3) solve $\cos y - x \sin y \frac{dy}{dx} = \cancel{\sec^2 x} \sec^2 x$

put $\cos y = u \Rightarrow -\sin y \frac{dy}{dx} = \frac{du}{dx}$

$$u + x\frac{du}{dx} = \sec^2 x$$

$\therefore \frac{du}{dx} + \frac{1}{x}u = \frac{\sec^2 x}{x}$, $P = \frac{1}{x}$, $Q = \frac{\sec^2 x}{x}$

$I.F. = e^{\int P dx} = x$

$\therefore u \cdot x = \int \frac{\sec^2 x}{x} \cdot x \, dx + c = \int \sec^2 x + c$

$\boxed{u \cdot \cos y = \tan x + c}$ is G.S.

(1) Solve $\frac{dy}{dx} = \frac{y^3}{e^{2x} + y^2}$

$\frac{e^{2x} + y^2}{y^3} = \frac{dx}{dy} \Rightarrow \frac{e^{2x}}{y^3} + \frac{1}{y} = \frac{dx}{dy}$

$\Rightarrow \frac{dx}{dy} - \frac{1}{y} = \frac{e^{2x}}{y^3}$

$\Rightarrow e^{-2x}\frac{dx}{dy} - \frac{e^{-2x}}{y} = \frac{1}{y^3}$

put $e^{+2x} = u$ $\therefore e^{-2x}\frac{dx}{dy} = -\frac{1}{2}\frac{du}{dy}$

$-\frac{1}{2} \cdot \frac{du}{dy} - \frac{u}{y} = \frac{1}{y^3}$ or $\frac{du}{dy} + \left(\frac{2}{y}\right)u = \frac{-2}{y^3}$

which is linear with $P = \frac{2}{y}$ & $Q = \frac{-2}{y^3}$

$IF = e^{\int \frac{2}{y} dy} = e^{2\log y} = y^2$

G.S. $u \cdot y^2 = \int -\frac{2}{y^3} \cdot y^2 dy + c = -2\int \frac{1}{y} dy + c$

$= -2\log y + c$

$\boxed{e^{-2x} \cdot y^2 = -2\log y + c}$ is G.S.

5) Solve $xy - \frac{dy}{dx} = y^3 e^{-x^2}$

Given diff. eqⁿ can be written as $\frac{dy}{dx} - xy = -y^3 e^{-x^2}$

This is Bernoulli's eqⁿ. Dividing by y^3

$y^{-3}\frac{dy}{dx} - x\cdot y^{-2} = -e^{-x^2}$, put $y^{-2} = u$

$\therefore -2y^{-3}\frac{dy}{dx} = \frac{du}{dx}$

$\therefore -\frac{1}{2}\frac{du}{dx} - x.u = -e^{-x^2}$ or $\frac{du}{dx} + (2x)u = 2e^{-x^2}$

$P = 2x, \quad Q = 2e^{-x^2}$

$\therefore IF = e^{\int 2x\,dx} = e^{x^2}$

$\therefore u e^{x^2} = \int 2 e^{-x^2}. e^{x^2}\, dx + c$

$= \int 2\, dx + c$

$\boxed{\frac{e^{x^2}}{y^2} = 2x + c}$ is the required G. S.

(6) Solve $xy + x^2y^3 = \frac{dx}{dy}$

Given $\frac{dx}{dy} - xy = x^2y^2$ or $x^{2}\frac{dx}{dy} - y x^{-1} = y^3$

put $x^{-1} = u$, $\therefore x^{-2}\frac{dx}{dy} = -\frac{du}{dy}$

$\therefore -\frac{du}{dy} - uy = y^3$ or $\frac{du}{dy} + y.u = -y^3$

$IF = e^{\int y\,dy} = e^{y^2/2}$

G.S. $u.e^{y^2/2} = \int -y^3 e^{y^2/2}\, dy + c$ put $y^2 = 2t$, $y\,dy = dt$

$= -\int e^t (2t)\, dt + c \quad = -2(t e^t - e^t) + c$

$= 2e^{y^2/2}\left(1 - \frac{y^2}{2}\right) c \quad = e^{y^2/2}(2 - y^2) + c$

$\boxed{\frac{e^{y^2/2}}{x} = e^{y^2/2}(2 - y^2) + c}$ is the required G. S.

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7) Solve $(xy^2 + e^{-1/x^3})dx - x^2y\,dy = 0$

Given $xy^2 + e^{-1/x^3} - x^2y \cdot \frac{dy}{dx} = 0$ or $y\frac{dy}{dx} - \frac{y^2}{x} = \frac{e^{-1/x^3}}{x^2}$

put $y^2 = u$ $\therefore y\frac{dy}{dx} = \frac{1}{2}\frac{du}{dx}$

$\therefore \frac{1}{2}\frac{du}{dx} - \frac{u}{x} = \frac{e^{-1/x^3}}{x^2}$ or $\boxed{\frac{du}{dx} - \frac{2}{x}u = \frac{2e^{-\frac{1}{x^3}}}{x^2}}$

G.S. is

$$u \cdot \frac{1}{x^2} = \int \frac{2e^{-1/x^3}}{x^2} \cdot \frac{1}{x^2}dx + c$$

put $\frac{-1}{x^3} = t$ $\therefore \frac{1}{x^4}dx = \frac{1}{3}dt$

$$\frac{u}{x^2} = \int \frac{2}{3}e^t dt + c = \frac{2}{3}e^t + c$$

$\boxed{\frac{3y^2}{x^2} - 2e^{-1/x^3} = c_1}$ is the G.S..

8) Solve $\cos x\frac{dy}{dx} + y\sin x = \sqrt{y\sec x}$

Given $\frac{dy}{dx} + y \cdot \tan x = \sqrt{y}\,\sec^{3/2} x$ (Bernoulli's D. Eqⁿ)

Divide by $\sqrt{y}$

$y^{-1/2}\frac{dy}{dx} + y^{1/2}\tan x = \sec^{3/2} x$. put $y^{1/2} = u$

$y^{-1/2}\frac{dy}{dx} = 2\frac{du}{dx}$

$2\frac{du}{dx} + u \cdot \tan x = \sec^{3/2} x$, $\frac{du}{dx} + \left(\frac{1}{2}\tan x\right)u = \frac{1}{2}\sec^{3/2} x$

$P = \frac{1}{2}\tan x$, $Q = \frac{1}{2}\sec^{3/2} x$

$\int P\,dx = \int \frac{1}{2}\cancel{\sec^{3/2} x}\,dx = \int\frac{1}{2}\tan x\,dx = \frac{1}{2}\log\sec x = \log\sqrt{\sec x}$

$\therefore$ I.F $= \sqrt{\sec x}$

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G.S. is $u\sqrt{\sec x} = \int \frac{1}{2}\sec^{\frac{3}{2}}x \cdot \sqrt{\sec x}\cdot dx + C$

$u\sqrt{\sec x} = \frac{1}{2}\int \sec^2 x\, dx + C = \frac{1}{2}\tan x + C$

$\boxed{\sqrt{y}\sec x = \frac{1}{2}\tan x + C}$ is the G.S.

① $xy(1+xy^2)\frac{dy}{dx} = 1$ Ans: $\frac{1}{x} = -y^2 + 2 + c e^{-y^2/2}$

② $\frac{dy}{dx} + \frac{1}{x}\tan y = \frac{1}{x^2}\tan y \cdot \sin y$ Ans: $\operatorname{cosec} y = \frac{1}{2x} + cx$

③ $e^y\left(1 + \frac{dy}{dx}\right) = e^x$ Ans $e^y \cdot e^x = \frac{e^{2x}}{2} + c$ (Hint put

4) $\frac{dy}{dx} + x\cdot \sin 2y = x^3\cos^2 y$ Ans: $\tan y = \frac{x^2-1}{2} + c e^{-x^2}$

⑤ $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$ Ans: $e^y = e^x - 1 + c e^{-e^x}$

⑥ $(\sec x \cdot \tan x \tan y - e^x)\, dx + \sec^2 y \cdot \sec x\, dy = 0$

Ans: $\tan y \cdot \sec x = e^x + c$

⑦ $\frac{dz}{dx} + \frac{z}{x}\log x = \frac{z}{x^2}(\log x)^2$ put $\log z = u$

Ans: $x\cdot \log z = \frac{(\log x)^3}{3} + c$

8) $3y^2\frac{dy}{dx} + 2xy^3 = 4xe^{-x^2}$ put $y^3 = u$, Ans: $y^3 e^{x^2} = 2x^2 + c$

9) $\frac{dy}{dx} + xy = y^2 e^{x^2/2}\log x$ Ans: $y^{-1}e^{-\frac{x^2}{2}} = -x\log x + x + c$

10) $x\frac{dy}{dx} + y = y^2\log x$ Ans: $\frac{1}{y} = cx + \log(ex)$

11) $xy^2\frac{dy}{dx} - y^3 = x^2$ Ans: $y^3 = x^2(cx - 3)$

12) $y\, dy = (x - y^2)\, dx$ Ans: $y^2 = x - \frac{1}{2} + c e^{-2x}$

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(13) $\frac{dy}{dx} - y.\tan x = y^4 . \sec x$ **Ans:** $y^{-3}\sec^3 x = -3\tan x - \tan^3 x + c$

(14) $y + 2\frac{dy}{dx} = y^3(x-1)$ **Ans:** $y^2(x + ce^x) = 1$

15) $2x\frac{dy}{dx} + y - 2x(x+1)y^3 = 0$ **Ans:** $1/xy^2 = -2(x + \log x) + c$

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(13) $\frac{dy}{dx} - y \cdot \tan x = y^4 \cdot \sec x$ Ans: $y^{-3} \sec^3 x = -3\tan x - \tan^3 x + c$

(14) $y + 2\frac{dy}{dx} = y^3(x-1)$ Ans: $y^2(x + c e^x) = 1$

15) $2x \frac{dy}{dx} + y - 2x(x+1) y^3 = 0$ Ans: $1/x y^2 = -2(x + \log x) + c$

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5] Exact differential eqn

(1)

consider a differential eqn of the form $M(x\,y)dx + N(x\,y)dy = 0$.

If there exist a function $u(x\,y)$ such that $Mdx + Ndy = du$ then the differential equation is called as an exact differential equation.

condition for exactness :

The necessary and sufficient condition that $Mdx + Ndy = 0$ be exact is

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

When the condition of exactness is satisfied, the general solution can be obtained by the following rule

Rule I :

$$\boxed{\int_{y=\text{const.}} Mdx + \int [\text{Terms of } N \text{ not containing } x]\, dy = c}$$

ie integrate Mdx w.r.t. x treating y constant, integrate only those terms in Ndy which are free from x w.r.t. y and equate their sum to constants.

Rule 2 : If N has no term which is free from x then

$$\boxed{\int_{y=\text{const}} Mdx = c}$$ is the general solution.

~~Rule 3 : If N has no term which is~~

Rule 3 : Sometimes we may write general solution by using the following rule

$$\boxed{\int_{x=\text{const}} Ndy + \int (\text{Term of } M \text{ not containing } y)\, dx = c}$$

Remarks: Sometimes an equation of the form $\dfrac{dy}{dx} = \dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ becomes exact if $\boxed{b_1 = -a_2}$ because the equation can be written as

$$(a_1x+b_1y+c_1)\,dx - (a_2x+b_2y+c_2)\,dy = 0$$

with $\dfrac{\partial M}{\partial y} = b_1$ & $\dfrac{\partial N}{\partial x} = -a_2$

accordingly the solution is given by

$$\boxed{\underset{\text{treat } y = \text{constant}}{\int} (a_1x+b_1y+c_1)\,dx - \int (b_2y+c_2)\,dy = c}$$

Ex 1 Solve $(x+y-2)\,dx + (x-y+4)\,dy = 0$ —①

This is of type $M\,dx + N\,dy = 0$ Here $M = x+y-2$ & $N = x-y+4$. $\dfrac{\partial M}{\partial y} = 1 = \dfrac{\partial N}{\partial x}$

eqn ① is exact diff. eqn. Its G.S. is given by

G.S. $\underset{y=\text{constant}}{\int} (x+y-2)\,dx + \underset{\text{Term free from } x \text{ in } N}{\int (-y+4)\,dy} = c$

$$\frac{x^2}{2} + xy - 2x - \frac{y^2}{2} + 4y = c$$

$$\boxed{x^2 - y^2 + 2xy - 4x + 8y = c}$$

Ex 2. Solve $\left(\dfrac{y^2}{(y-x)^2} - \dfrac{1}{x}\right)dx + \left(\dfrac{1}{y} - \dfrac{x^2}{(x-y)^2}\right)dy = 0$

Here $M = \dfrac{y^2}{(y-x)^2} - \dfrac{1}{x}$ $\therefore \dfrac{\partial M}{\partial y} = \dfrac{2y}{(y-x)^2} - \dfrac{2y^2}{(y-x)^3} = \dfrac{-2xy}{(y-x)^3}$

& $N = \dfrac{1}{y} - \dfrac{x^2}{(x-y)^2}$ $\therefore \dfrac{\partial N}{\partial x} = \dfrac{-2x}{(x-y)^2} + \dfrac{2x^2}{(x-y)^3} = \dfrac{-2xy}{(y-x)^3}$

Therefore the given equation is exact

$\therefore$ G.S. is $\quad y^2 \int \frac{1}{(y-x)^2}dx - \int \frac{1}{x}dx + \int \frac{1}{y}dy = c$

ie $\quad y^2 \cdot \frac{1}{y-x} - \log x + \log y = c$

$$\boxed{\frac{y^2}{y-x} + \log \frac{y}{x} = c}$$ is G.S.

(3) solve $\quad \frac{dy}{dx} = \frac{\tan y - 2xy - y}{x^2 - x\tan^2 y + \sec^2 y}$

given diff. eq^n. can be written as

$(\tan y - 2xy - y)dx - (x^2 - x\tan^2 y + \sec^2 y)dy = 0$

$M = \tan y - 2xy - y \qquad N = x\tan^2 y - x^2 - \sec^2 y$

$\frac{\partial M}{\partial y} = \sec^2 y - 2x - 1 \qquad \frac{\partial N}{\partial x} = \tan^2 y - 2x$

$= \tan^2 y - 2x \qquad = \tan^2 y - 2x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ give eq^n is exact. diff-eq^n.

G.S. is $\quad \int\limits_{y=\text{const.}} (\tan y - 2xy - y)\,dx + \int\limits_{(\text{No. } x)} -\sec^2 y\,dy = c$

$\Rightarrow \boxed{x\tan y - x^2 y - xy - \tan y = c}$ is G.S.

(4) Solve $\quad \left(\frac{y}{(x-y)^2} - \frac{1}{2\sqrt{1-x^2}}\right)dx - \frac{x}{(x-y)^2}dy = 0$

Here $\quad M = \frac{y}{(x-y)^2} - \frac{1}{2\sqrt{1-x^2}} \quad$ & $\quad N = \frac{-x}{(x-y)^2}$

$$\frac{\partial M}{\partial y} = \frac{1}{(x-y)^2} + \frac{2y}{(x-y)^3} = \frac{x+y}{(x-y)^3}$$

$$\& \quad \frac{\partial N}{\partial x} = \frac{-1}{(x-y)^2} + \frac{2x}{(x-y)^3} = \frac{x+y}{(x-y)^3}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Given differential eqn is exact and the G.S. is given by

$$y\int \left[\frac{1}{(x-y)^2}\right] dx - \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\, dx = c$$

$$\Rightarrow \frac{-y}{x-y} - \frac{1}{2}\sin^{-1}x = c$$

$$\Rightarrow \boxed{\frac{2y}{x-y} + \sin^{-1}x = c_1}$$ is the G.S.

(5) Solve $(y^2 e^{xy^2} + 4x^3)\,dx + (2xy\, e^{xy^2} - 3y^2)\,dy = 0$

Here $M = y^2 e^{xy^2} + 4x^3 \qquad N = 2xy\, e^{xy^2} - 3y^2$

$$\frac{\partial M}{\partial y} = 2y\, e^{xy^2} + y^2 e^{xy^2}(2xy) \;,\; \frac{\partial N}{\partial x} = 2y\, e^{xy^2} + 2xy\, e^{xy^2}(y^2)$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ Given diff. eqn is exact. & G.S. is

$$\int_{y=\text{constant}} (y^2 e^{xy^2} + 4x^3)\,dx + \int (-3y^2)\,dy = c$$

$$\boxed{y^2 \cdot \frac{e^{xy^2}}{y^2} + x^4 - y^3 = c} \Rightarrow e^{xy^2} + x^4 - y^3 = c$$

Ex (6) $\left(\frac{2x}{y^3}\right)dx + \left(\frac{y^2 - 3x^2}{y^4}\right)dy = 0$ Ans: $\frac{x^2}{y^3} - \frac{1}{y} = c$

(7) $\frac{dy}{dx} = \frac{2x-3y+1}{3x+4y-5}$ Ans: $x^2 - 3xy + x - 2y^2 + 5y = c$

8) $\frac{dy}{dx} = -\frac{4x^3y^2 + y\cos xy}{2x^4y + x\cos xy}$ Ans: $x^4y^2 + \sin xy = c$.

9) $\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}$ Ans: $x^2 - 4xy + 10x - y^2 + 2y = c$

10) $\frac{1}{2x}\frac{dy}{dx} + \frac{x+y}{x^2+y^2} = 0$. Ans: $2x^3 + 3x^2y + y^3 = c$

11) $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x}$ Ans: $y\sin x + x\sin y + xy = c$

12) $[y(1+\frac{1}{x}) + \cos y]\,dx + [x + \log x - x\sin y]\,dy = 0$

Ans: $y(x + \log x) + x\cos y = c$

13) $(x\sqrt{1-x^2y^2} - y)\,dy + (x + y\sqrt{1-x^2y^2})\,dx = 0$

Ans: $x^2 + xy\sqrt{1-x^2y^2} + \sin^{-1}(xy) = c$.

14) $(1 + xy^2)\,dx + (1 + x^2y)\,dy = 0$. Ans: $x + \frac{x^2y^2}{2} + y = c$

15. $(2xy^4 + \sin y)\,dx + (4x^2y^3 + x\cos y)\,dy = 0$

Ans: $x^2y^4 + x\sin y = c$.

16. $\frac{dy}{dx} = \frac{5-3x-2y}{2x+3y-5}$ Ans: $4xy + 3(x^2+y^2) - 10(x+y) = c$

17 $\frac{dy}{dx} = \frac{4x-2y+1}{2x-6y+2}$ Ans: $2xy + 2y - 3y^2 - x - 2x^2 = c$

18 $(1+x^2)(x\,dy + y\,dx) = -2yx^2\,dx$ Ans: $xy(1+x^2) = c$.

Equations Reducible to exact by using integrating factor

Integrating factor is a multiplying factor by which the equation can be made exact.

Rules for finding integrating factors of the equation $Mdx + Ndy = 0$ when it is not exact.

Rule I : If $xM + yN \neq 0$, and the given equation (diff. eqn) is homogeneous then

$$\boxed{I.F = \frac{1}{xM + yN}}$$

Rule II : If $xM - yN \neq 0$ and given diff. eqn has the form

$y \cdot f_1(xy)\,dx + x f_2(xy)\,dy = 0$ then $\boxed{I.F. = \frac{1}{xM - yN}}$

Rule III : If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ (say) then $\boxed{IF = e^{\int f(x)\,dx}}$

Rule IV : If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \phi(y)$ (say.) then ~~IF~~ =

$$\boxed{I.F = e^{\int \phi(y)\,dy}}$$

Rule V : If the equation $Mdx + Ndy = 0$ can be written as

$$x^a y^b (my\,dx + nx\,dy) + x^r y^s (py\,dx + qx\,dy) = 0$$

Where, a, b, m, n, r, s, p and q are the constants having any value, then $I.F = x^h y^k$. Where h & k are such that after multiplying the integrating factor, the condition of exactness is satisfied.

Illustrations on Eqn Reducible to exact form

Rule I

Solve ① $(xy-2y^2)dx-(x^2-3xy)dy=0$

Here $M = xy-2y^2$ $\qquad N = -x^2+3xy$

$\frac{\partial M}{\partial y} = x-4y$ and $\frac{\partial N}{\partial x} = -2x+3y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ ∴ Eqn is not exact but it is homogeneous

Also $xM+yN = x(xy-2y^2)+y(-x^2+3xy)$

$= x^2y-2xy^2-x^2y+3xy^2 = xy^2 \neq 0$

By using Rule I $\quad I.F = \frac{1}{xM+yN} = \frac{1}{xy^2}$

multiply given eqn by $\frac{1}{xy^2}$, we get

$$\left(\frac{1}{y}-\frac{2}{x}\right)dx+\left(-\frac{x}{y^2}+\frac{3}{y}\right)dy=0$$

which is exact. Its G.S. is

$$\boxed{\frac{x}{y}-2\log x+3\log y=c}$$

② $(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$

Here $M = x^2y-2xy^2$ $\qquad N = -x^3+3x^2y$

Here $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ Given diff. eqn is not exact. But it is Homogeneous diff eqn

$$\therefore xM + yN = x(x^2y - 2xy^2) + y(-x^3 + 3x^2y)$$
$$= x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2 \neq 0$$

$$\therefore \text{I.F.} = \frac{1}{xM + yN} = \frac{1}{x^2y^2}$$

multiply given eqn by I.F. we get

$$\frac{1}{x^2y^2}(x^2y - 2xy^2)dx - \frac{1}{x^2y^2}(x^3 - 3x^2y)dy = 0$$

$$\left[\frac{1}{y} - \frac{2}{x}\right] - \left(\frac{x}{y^2} + \frac{3}{y}\right)dy = 0$$

which is exact. diff. eqn & its Solution is given by

$$\boxed{\log y - 2\log x + 3\log y = c}$$

(3) $(3xy^2 - y^3)dx + (xy^2 - 2x^2y)dy = 0$ Ans: $\text{I.F} = \frac{1}{x^2y^2}$

$cy^2 = x^3 e^{y/x}$

(4) $(x^2 - 3xy + 2y^2)dx + x(3x - 2y)dy = 0$

$\text{I.F.} = \frac{1}{x^3}$, Ans: $x^2\log x + 3xy - y^2 = cx^2$

(5) $x(x-y)\frac{dy}{dx} = y(x+y)$ $\left(\text{I.F.} = \frac{1}{2xy^2}\right)$ Ans: $\boxed{xy^2 = c}$

Rule II:

Ex (1) Solve $(x^2y^2 + 2)y\,dx + (2 - 2x^2y^2)x\,dy = 0$

Here $\frac{\partial M}{\partial y} = 3x^2y^2 + 2$, $\frac{\partial N}{\partial x} = 2 - 6x^2y^2$

It can be seen that given eqn is not exact. It is in the form

$f_1(xy)\, y\, dx + f_2(xy)\, x\, dy = 0$ (by using rule 2)

$\therefore$ I.F $= \dfrac{1}{xM - yN} = \dfrac{1}{x^3y^3 + 2xy - 2xy + 2x^3y^3} = \dfrac{1}{3x^3y^3}$

Now multiply by $\dfrac{1}{3x^3y^3}$ to given eqn, we get

$$\frac{1}{3x^3y^3}\left[x^2y^3 + 2y\right]dx + \frac{1}{3x^3y^3}\left[2x - 2x^3y^2\right] = 0$$

$$\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx + \left(\frac{2}{3x^2y^3} - \frac{2}{3y}\right)dy = 0$$

Which is exact. Its G.S. is

$$\int \frac{1}{3x}dx + \frac{2}{3y^2}\int \frac{1}{x^3}dx - \frac{2}{3}\int \frac{1}{y}dy = c$$

$$\Rightarrow \log x - \frac{1}{y^2x^2} - 2\log y = c_1 \Rightarrow \boxed{\log\left(\frac{x}{y^2}\right) - \frac{1}{x^2y^2} = c}$$

(2) $y(1+xy)dx + x(1+xy+x^2y^2)dy = 0.$

Given eqn is not exact diff-eqn and it can written in the form.

$$y\, f_1(xy)\, dx + x\, f_2(xy)\, dy = 0$$

& $xM - yN = xy + x^2y^2 + x^2y - x^2y^2 - x^3y^3$

$= -x^3y^2 \neq 0$

$\therefore$ I.F. $= \dfrac{1}{xM - yN} = \dfrac{-1}{x^3y^3}.$

multiplying given by eqn by $-\dfrac{1}{x^3y^3}$ We get

$$\left(-\frac{1}{x^3y^2}-\frac{1}{x^2y}\right)dx+\left(\frac{-1}{x^2y^3}-\frac{1}{xy^2}-\frac{1}{y}\right)dy=0 \quad -(1)$$

for eqn (1) $\frac{\partial M}{\partial y}=\frac{2}{x^3y^3}+\frac{1}{x^2y^2}=\frac{\partial N}{\partial x}$

so eqn (1) is exact and its G.S. is given by

$$\int\left(-\frac{1}{x^3y^2}-\frac{1}{x^2y}\right)dx+\int-\frac{1}{y}\,dy=c$$

$$-\frac{1}{y^2}\int\frac{1}{x^3}dx-\frac{1}{y}\int\frac{1}{x^2}dx-\int\frac{1}{y}dy=c$$

$\boxed{\frac{1}{2y^2x^2}+\frac{1}{xy}-\log y=c}$ is G.S.

(3) $(x^2y^2+xy+1)y\,dx+(x^2y^2-xy+1)x\,dy=0.$

$\boxed{I.F=\frac{1}{2x^2y^2}}$ Ans: $xy+\log x-\frac{1}{xy}-\log y=c$

(4) $(x^2y^2+5xy+2)y\,dx+(x^2y^2+4xy+2)x\,dy=0$, $\boxed{I.F=\frac{1}{x^2y^2}}$

Ans: $xy+5\log x+4\log y-\frac{2}{xy}=c$

(5) $(x^3y^3+x^2y^2+xy+1)y\,dx+(x^3y^3+x^2y^2-xy-1)x\,dy=0$ $\boxed{I.F=\frac{1}{2xy}}$

Ans: $\frac{x^2y^2}{2}+\log\frac{x}{y}=c$

(6) $(1+xy)y\,dx+(1-xy)x\,dy=0$ $I.F=\frac{1}{2x^2y^2}$

Ans: $\boxed{\log\frac{x}{y}-\frac{1}{xy}=c}$

(7) $(xy\sin xy+\cos xy)y\,dx+(xy\sin xy-\cos xy)x\,dy=0.$

$I.F=\frac{1}{2xy\cos xy}$, Ans: $\boxed{y\cos xy=cx}$

$\boxed{\text{Rule 3}}$ & $\boxed{\text{Rule 4}}$

Solve (1) $(x^2+y^2+x)\,dx + (xy)\,dy = 0$

Here $M = x^2+y^2+x$ $\qquad N = xy$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = y$

To find I.F. consider $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-y}{xy} = \frac{1}{x} = f(x)$

$\therefore$ I.F $= e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = x.$

Now multiplying by I.F $= x$ to eqn (1), we get

$$(x^3+xy^2+x^2)\,dx + (x^2y)\,dy = 0 \qquad -(2)$$

Thus eqn (2) is exact, its G.S. is given by

G.S. $\int (x^3+xy^2+x^2)\,dx = c$

$\Rightarrow \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = c$

$\boxed{3x^4+6x^2y^2+4x^3 = c_1}$ is the G.S.

(2) Solve $(x^4e^x - 2mxy^2)\,dx + (2mx^2y)\,dy = 0 \qquad -(1)$

Given eqn is of the type $M\,dx + N\,dy = 0$

$M = x^4e^x - 2mxy^2 \qquad N = 2mx^2y$

$\frac{\partial M}{\partial y} = -4mxy \qquad \frac{\partial N}{\partial x} = 4mxy$

eqn (1) is not exact. To find I.F. (by rule 3) we have

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -8mxy$

$$I.F = e^{-4\int \frac{1}{x}dx} = e^{-4\log x} = \frac{1}{x^4}.$$

Now multiplying to eqn ① by $\frac{1}{x^4}$ We get

$$\left(e^x - \frac{2my^2}{x^3}\right)dx + \left(\frac{2my}{x^2}\right)dy = 0 \quad —②$$

eqn ② is exact, Its G.S. is

$$\int \left(e^x - \frac{2my^2}{x^3}\right)dx = c \Rightarrow \boxed{e^x + \frac{my^2}{x^2} = c}$$

③ Solve $y(2x^2y + e^x)\,dx = (e^x + y^3)\,dy$

Given $(2x^2y^2 + ye^x)\,dx + (-e^x - y^3)\,dy = 0 \quad —①$

Here $\frac{\partial M}{\partial y} = 4x^2y + e^x$, $\frac{\partial N}{\partial x} = -e^x$ ∴ ① is not exact

By Rule ④

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-e^x - 4x^2y - e^x}{y(2x^2y + e^x)} = \frac{-2(2x^2y + e^x)}{y(2x^2y + e^x)}$$

$$= \frac{-2}{y}$$

$$I.F = e^{\int -\frac{2}{y}dy} = \frac{1}{y^2}$$

∴ eqn ① become

$$\left(2x^2 + \frac{e^x}{y}\right)dx + \left(-\frac{e^x}{y^2} - y\right)dy = c \quad —②$$

eqn ② is exact and its General soln is given by

$$\int \left(2x^2 + \frac{e^x}{y}\right)dx - \int y\,dy = c$$

$$\Rightarrow 2x^3 + \frac{e^x}{y} - \frac{y^2}{2} = c \text{ is G.S.}$$

(4) Solve $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)dx + \left(\frac{x+xy^2}{4}\right)dy = 0$

Here $M = y + \frac{y^3}{3} + \frac{x^2}{2}$ $\quad N = \frac{x+xy^2}{4}$

$\frac{\partial M}{\partial y} = 1+y^2 \quad \frac{\partial N}{\partial x} = \frac{1+y^2}{4}$

Consider $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{(1+y^2) - \left(\frac{1+y^2}{4}\right)}{x\left(\frac{1+y^2}{4}\right)} = \frac{3}{x} = f(x)$

$\therefore$ I.F. $= e^{\int f(x)dx} = e^{\int \frac{3}{x}dx} = e^{3\log x} = x^3$

multiplying by x^3 to the given eqn,

$$\left(yx^3 + \frac{x^3y^3}{3} + \frac{x^5}{2}\right)dx + \left(\frac{x^4+x^4y^2}{4}\right)dy = 0 \quad \text{--- (1)}$$

the eqn (1) is exact and its G.S is

G.S. $y\int(x^3dx) + y^3\int\frac{x^3}{3}dx + \frac{1}{2}\int x^5 dx = c$

$$y\frac{x^4}{4} + y^3\frac{x^4}{3\times4} + \frac{1}{2}\frac{x^6}{6} = c$$

$\boxed{3yx^4 + x^4y^3 + x^6 = c}$ is the G.S.

(5) Solve $(x\sec^2 y - x^2\cos y)\,dy = (\tan y - 3x^4)\,dx$

Ans: $\frac{\tan y}{x} + x^3 - \sin y = c$, & IF $= \frac{1}{x^2}$

(Rule3)

(6) solve $(y^4+2y)\,dx + (xy^3+2y^4-4x)\,dy = 0$ (Rule4)

I.f. $= \frac{1}{y^3}$, Ans: $\left(y+\frac{2}{y^2}\right)x + y^2 = c$.

Date : / /

Rule ③ & ④ Examples (Exercise)

① $(2x\log x - xy)\,dy + 2y\,dx = 0$. $(IF = \frac{1}{x})$. Ans: $2y\log x - \frac{y^2}{2} = c$

② $(y - 2x^3)\,dx - x(1 - xy)\,dy = 0$ $(IF = \frac{1}{x^2})$, Ans $\frac{y}{x} + x^2 - \frac{y^2}{2} = c$

③ $(x^4 e^x - 2my^2)\,dx + 2mx^2 y\,dy = 0$ $(IF = \frac{1}{x^4})$ Ans: $e^x + \frac{my^2}{x^2} = c$

④ $(x - y^2)\,dx + 2xy\,dy = 0$ $(I.F. = \frac{1}{x^2})$ Ans: $\frac{y^2}{x} + \log x = c$

⑤ $(x^2 + y^2 + 1)\,dx - 2xy\,dy = 0$ $(I.F = \frac{1}{x^2})$ Ans: $x^2 - y^2 - 1 = cx$

Rule 4

⑥ $[3x^2y^4 + 2xy]\,dx + (2x^3y^3 - x^2)\,dy = 0$ $(I.F = \frac{1}{y^2})$ Ans: $x^3y^2 + \frac{x^2}{y} = c$

⑦ $y(x^2y + e^x)\,dx - e^x\,dy = 0$ $(I.F = \frac{1}{y^2})$ Ans: $\frac{x^3}{3} + \frac{e^x}{y} = c$

⑧ $(2x + e^x\log y)\,y\,dx + e^x\,dy = 0$ $(IF = \frac{1}{y})$ Ans: $x^2 + e^x\log y = c$

⑨ $(\frac{y}{x}\sec y - \tan y)\,dx = (x - \sec y\log x)\,dy$ $(I.F = \frac{1}{\sec y})$

Ans: $y\log x - x\sin y = c$

⑩ $y\log y\,dx + (x - \log y)\,dy = 0$ $(IF = \frac{1}{y})$

Ans: $2x\log y - (\log y)^2 = c$

⑪ $y(2xy + e^x)\,dx = e^x\,dy$ $(IF) = \frac{1}{y^2}$ Ans: $x^2 + \frac{e^x}{y} = c$